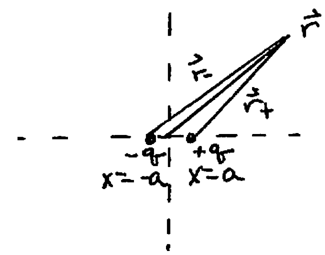
Electrocardiography analysis: Potential differences in an electric dipole's field

The goal of this analysis is to relate measured potential differences on the surface of the body to the components of the heart's equivalent[1] electric dipole moment.

Consider an electric dipole consisting of charges ±q at $x = a$ and $x = -a$. The potential produced by the dipole at any location $\vec{r}$ is given by adding the potentials of the individual charges:



$$V_{dipole}(\vec{r}) = V_{+q}(\vec{r}_+) + V_{-q}(\vec{r}_-) = \frac{q}{4\pi\varepsilon_0 r_+} + \frac{-q}{4\pi\varepsilon_0 r_-} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

(using $1/4\pi\varepsilon_0$ rather than k to avoid confusion with the dielectric constant)

If we embed this dipole in a medium with dielectric constant κ, the potential, like the electric field, is reduced by a factor of κ:

$$V_{dipole}(\vec{r}) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

Let us find the potential difference between two points on the y-axis. At any location $\vec{r}$ on the y-axis, $V_{dipole}(0,y,0) = 0$ because $r_+ = r_-$. So the potential is constant along the y-axis and the potential difference between any two points on the y-axis is zero.

Now consider finding the potential difference between two points on the x-axis that are equal distances r from the origin (at x = r and x = -r). The potential at x = r is given by

$$V_{dipole}(r,0,0) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r-a} - \frac{1}{r+a}\right) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{r+a-(r-a)}{(r-a)(r+a)}\right) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{2a}{r^2-a^2}\right)$$

Using the definition of the magnitude of the dipole moment $p = qd = 2qa$, in the case where $r >> a$, the denominator of the fraction can be approximated as r^2 and this can be written as

$$V_{dipole}(r,0,0) = \frac{p}{4\pi\varepsilon_0\kappa r^2}$$

The same procedure gives the potential at x = -r as

$$V_{dipole}(-r,0,0) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r+a} - \frac{1}{r-a}\right) = -\frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r-a} - \frac{1}{r+a}\right) \approx -\frac{p}{4\pi\varepsilon_0\kappa r^2}$$

The potential difference between $x = +r$ and $x = -r$ is therefore

$$\Delta V = V_{dipole}(r,0,0) - V_{dipole}(-r,0,0) = \frac{2p}{4\pi\varepsilon_0\kappa r^2}$$

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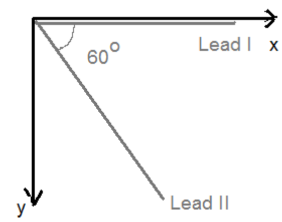
[1] As also discussed in the lab writeup, the heart's dipole moment is actually a current dipole rather than a charge dipole, so we call this the "equivalent electric dipole moment" because what we finally calculate here is the charge dipole moment that would produce the same electric field that the heart's current dipole produces.

A dipole in this orientation corresponds to a dipole moment on the x-axis in the positive x-direction. The heart's dipole moment does not have to be in the x-direction (and in fact it changes direction during the cardiac cycle). However, we can write its dipole moment as the sum of an x-component and a y-component. For a dipole moment on the y-axis, everything we worked out is true switching x and y, so now the x-axis is an equipotential and the potential difference we found above is for points at y = ±r on the y-axis.

Here is the really useful (and cool) point: **Because the axis perpendicular to the dipole is an equipotential, the potential difference between two points on the x-axis depends only on the x-component of the dipole p_x, and the potential difference between two points on the y-axis depends only on p_y.** So we can measure ΔV between two points on a horizontal line (Lead I) that are equidistant from the dipole to determine p_x, and we can measure ΔV between two points on a vertical line equidistant from the dipole to determine p_y.

Due to the way the electric field spreads through the body, it turns out that effectively Lead II is located at 60° to Lead I and only corresponds to half of the horizontal distance of Lead I. Consequently, the potential difference across lead 2 corresponds to findng the potential difference along a horizontal path, and then adding the potential difference along a vertical path as shown in the figure. Therefore, to find just ΔV along the vertical path, we can calculate it from Lead II – (1/2) Lead I.



We can thus calculate the components of the equivalent dipole moment from the horizontal and vertical potential differences:

Horizontal: $$\Delta V_{\text{horiz}} = \Delta V_{\text{Lead I}} = V(r,0,0) - V(-r,0,0) = \frac{p_x}{2\pi\varepsilon_0\kappa_{water}r^2}$$

Vertical: $$\Delta V_{\text{vert}} = \Delta V_{\text{Lead II}} - \frac{1}{2}\Delta V_{\text{Lead I}} = V(0,r,0) - V(0,-r,0) = \frac{p_y}{2\pi\varepsilon_0\kappa_{water}r^2}$$

This pair of equations is true at every instant in time.

Obviously there are a lot of approximations involved here! Probably the most significant are that the heart is not centered between the electrodes (especially for Lead II) and the effective angle between Leads I and II will vary somewhat patient by patient due to the patient's anatomy. The clinical electrocardiogram uses twelve leads to obtain a more comprehensive picture of the three-dimensional behavior. We are doing this simplified analysis so that you understand the basic physics and get a feel for the complex fields generated by the beating heart. If any of you go on to become cardiologists, you'll find that there are more sophisticated ways to analyze the data.

These same results can be obtained using the expression derived by Wolfson in Ex. 22.5 which describes the position in terms of an angle instead of x-y coordinates:

$$V_{dipole}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^2}\cos\theta$$

In this case, points on the y-axis correspond to θ = ±90°; points on the positive x-axis correspond to θ = 0° and points on the negative x-axis correspond to θ = 180°.