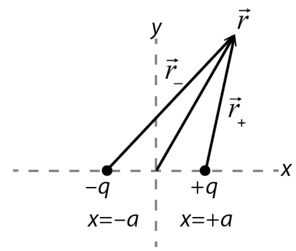
Electrocardiography Laboratory Appendix: Calculating Potential Differences in a Dipole Field

Swarthmore College Introductory Physics for the Life Sciences

The goal of this analysis is to relate measured potential differences on the surface of the body to the components of the heart's electric dipole moment.

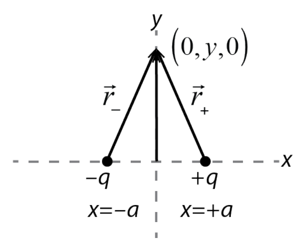
Consider an electric dipole consisting of charges $\pm q$ at $x = a$ and $x = -a$. The potential of the dipole at any location $\vec{r}$ can be found by adding the potentials of the individual charges:



$$V_{dipole}(\vec{r}) = V_{+q}(\vec{r}_+) + V_{-q}(\vec{r}_-) = \frac{q}{4\pi\varepsilon_0 r_+} + \frac{-q}{4\pi\varepsilon_0 r_-} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{A.1}$$

If we embed this dipole in a dielectric with dielectric constant κ, the potential, like the electric field, is reduced by a factor of κ:

$$V_{dipole}(\vec{r}) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{A.2}$$



In the dipole investigation at the beginning of lab, you found from an electric field diagram that for a horizontal dipole, V_{dipole} is constant along the y-axis. You can now confirm this mathematically. Apply (A.2) to any location $\vec{r} = (0, y, 0)$ on the y-axis; $r_+ = r_- = \sqrt{a^2 + y^2}$, as shown in the figure, and consequently $V_{dipole}(0, y, 0) = 0$ at any y. As a result, $\Delta V_{y_1 y_2}$ produced by this dipole between any locations y_1 and y_2 on the y-axis is zero:[1]

$$\Delta V_{y_1 y_2} = V_{dipole}(0, y_2, 0) - V_{dipole}(0, y_1, 0) = 0. \tag{A.3}$$

Now consider finding the potential difference between two points on the x-axis that are equal distances r from the origin (at $x = r$ and $x = -r$). The potential at $x = r$ is given by

$$V_{dipole}(r, 0, 0) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r-a} - \frac{1}{r+a}\right) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{r+a-(r-a)}{(r-a)(r+a)}\right) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{2a}{r^2 - a^2}\right) \tag{A.4}$$

Using the definition of the magnitude of the dipole moment $p = qd = 2qa$, in the case $r \gg a$, the denominator of the fraction can be approximated as r^2 and this can be written as

$$V_{dipole}(r, 0, 0) = \frac{p}{4\pi\varepsilon_0\kappa r^2} \tag{A.5}$$

The same procedure gives the potential at $x = -r$ as

$$V_{dipole}(-r, 0, 0) = \frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r+a} - \frac{1}{r-a}\right) = -\frac{q}{4\pi\varepsilon_0\kappa}\left(\frac{1}{r-a} - \frac{1}{r+a}\right) \approx -\frac{p}{4\pi\varepsilon_0\kappa r^2} \tag{A.6}$$

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[1] Clarification of a possibly confusing point: Not only is V_{dipole} *constant*, its *value* happens to be zero; what matters is that V_{dipole} has the same value for any y, so that the potential *difference* between any two points on the y-axis is zero.

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The potential difference between $x = +r$ and $x = -r$ is therefore

$$\Delta V = V_{dipole}(r,0,0) - V_{dipole}(-r,0,0) = \frac{2p}{4\pi\varepsilon_0 \kappa r^2} \tag{A.7}$$

The dipole moment of a dipole constructed this way points in the positive x-direction. The heart's dipole moment changes direction during the cardiac cycle, so this is not a complete model of the heart's dipole moment. However, the heart's dipole moment $\vec{p}_{heart}$ can always be written as the sum of x- and y-components $p_{heart,x}$ and $p_{heart,y}$. For a dipole with its dipole moment in the y-direction, everything we worked out above applies with x and y switched, so now the x-axis is an equipotential and (A.7) gives the potential difference between locations $y = \pm r$ on the y-axis.

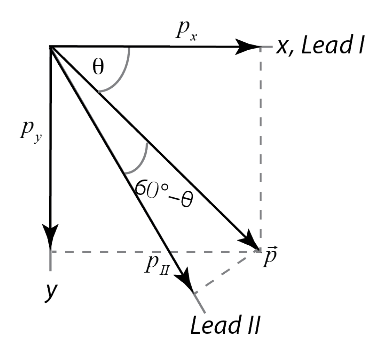
Here is the really useful (and cool) point: For any electric dipole, because the axis perpendicular to the dipole is an equipotential, the potential difference between two points on the x-axis depends only on the x-component of the dipole p_x, and the potential difference between two points on the y-axis depends only on p_y. So, in an electrocardiogram, we can measure ΔV between two points on a horizontal line (Lead I) to determine $p_{heart,x}$, and we can measure ΔV between two points on a vertical line to determine $p_{heart,y}$. In other words, if ΔV_{horiz} is measured between $\pm x$ and if ΔV_{vert} is measured between $\pm y$:

$$\Delta V_{\text{horiz}} = \frac{p_x}{2\pi\varepsilon_0 \kappa_{water} x^2} \text{ and } \Delta V_{\text{vert}} = \frac{p_y}{2\pi\varepsilon_0 \kappa_{water} y^2} \tag{A.8}$$

which can be solved to give

$$p_x = 2\pi\varepsilon_0 \kappa_{water} x^2 \Delta V_{\text{horiz}} \text{ and } p_y = 2\pi\varepsilon_0 \kappa_{water} y^2 \Delta V_{\text{vert}}. \tag{A.9}$$

It turns out that the complexities of human physiology make it difficult to directly measure a potential difference along a vertical line. Due to the way the electric field spreads through the body, it turns out that effectively Lead II is located at 60° to Lead I. Consequently, we use trigonometry to calculate $p_{heart,y}$ from the Lead I and Lead II measurements as follows.



Consider a dipole oriented at an arbitrary angle θ as shown. The horizontal and vertical components p_x and p_y are given by

$$p_x = p\cos\theta \text{ and } p_y = p\sin\theta. \tag{A.10}$$

The component of the dipole along Lead II, which we can notate p_{II}, is given by

$$p_{II} = p\cos(60° - \theta) \tag{A.11}$$

which can be simplified using an angle difference formula to

$$p_{II} = p\cos 60° \cos\theta + p\sin 60° \sin\theta = \frac{1}{2}p\cos\theta + \frac{\sqrt{3}}{2}p\sin\theta = \frac{1}{2}p_x + \frac{\sqrt{3}}{2}p_y \tag{A.12}$$

We can then solve for p_y in terms of p_x and p_{II}:

$$p_{II} - \frac{1}{2}p_x = \frac{\sqrt{3}}{2}p_y \quad \rightarrow \quad p_y = \frac{2}{\sqrt{3}}\left(p_{II} - \frac{1}{2}p_x\right) \cong 1.15\left(p_{II} - \frac{1}{2}p_x\right) \tag{A.13}$$

Finally, we note that the relationship we found above between p_x and p_y and the voltages measured along the corresponding axes is true for any combination of a dipole component and the voltage measured along the corresponding axis. So, it is also true that the component of the heart's dipole along Lead II is related to the Lead II voltage by

$$p_{II} = 2\pi\varepsilon_0\kappa_{water}r_{II}^2\Delta V_{\text{Lead II}} \tag{A.14}$$

If the distances at which the leads are all measured are roughly equal, $x \cong y \cong r_{II} \cong r$, we can approximate the components of the dipole moment from:

Horizontal: $p_x = 2\pi\varepsilon_0\kappa_{water}r^2\Delta V_{\text{Lead I}}$

Vertical: $p_y \cong 1.15\left(p_{II} - \frac{1}{2}p_x\right) = 2\pi\varepsilon_0\kappa_{water}r^2 1.15\left(\Delta V_{\text{Lead II}} - \frac{1}{2}\Delta V_{\text{Lead I}}\right)$

This pair of equations is true at every instant in time. Within the limits of the approximation that the distances are all the same, the proportion between any component of the dipole moment and the corresponding voltage is the same. Therefore, to plot the time-dependent dipole moment, you can do the calculations and plotting just using the voltages.

To find the maximum dipole moment, find the peak signal in whichever lead shows the strongest signal, and then find the value of the other lead at that same instant of time. Then, use the corresponding ΔV_{horiz} and ΔV_{vert} to find the maximum dipole moment.

There are a lot of approximations involved here! Probably the most significant is that the effective angle between Leads I and II will vary somewhat patient by patient due to the patient's anatomy; the clinical electrocardiogram uses twelve leads to obtain a more comprehensive picture of the three-dimensional behavior. We are doing this simplified analysis so that you understand the basic physics and get a feel for the complex fields generated by the beating heart. If you go on to become a cardiologist, you'll find that there are more sophisticated and hence more accurate ways to analyze the data.